# Solving Trig Equations with the Pythagorean Identities (3.12) HW

For questions #1-6, find the solutions in the interval $0 \leq \theta < 2\pi$. Then, find an expression that represents ALL solutions.

1. $\csc^2\theta + \csc\theta - 2 = 0$
2. $2\sin^2\theta = 2 + \cos\theta$
3. $\sec^2\theta + 2\tan\theta = 0$
4. $2\cos^2\theta + 3\sin\theta - 3 = 0$
5. $\csc^2\theta + \cot\theta = 1$
6. $\sqrt{2}\cos\theta\sin\theta = \sin\theta$

In questions #8 and 9, write an expression that represents all zero(s) of the given function.

7. $f(x) = \sin^2 x + \cos x + 1$
8. $g(\theta) = 1 + \tan^2\theta$

9. Let $f(x) = sec^2\,x$. Find an expression for all input values that yield an output value of $1$ for this function.

10. Which one of the following is equivalent to $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2$?
    A) $1$
    B) $2$
    C) $-2\sin\theta\cos\theta$
    D) $4\sin\theta\cos\theta$

11. Which one of the following is equivalent to $(2 + \tan\theta)^2 + (1 - 2\tan\theta)^2$?
    A) $5$
    B) $-5\tan\theta$
    C) $5\sec^2\theta$
    D) $-5\csc^2\theta$

12. Which one of the following is equivalent to $(2 - \csc\theta)(2 + \csc\theta)$?
    A) $5 + \cot^2\theta$
    B) $3 - \cot^2\theta$
    C) $-4\cot^2\theta$
    D) $\cot^2\theta - 5$

The way we write our answer when asked for "all solutions" is weird. The point of #13 and #14 is to make sure we understand how to interpret these answers.

13. Sharon solves a trigonometric equation and states "All solutions are of the form $\theta = \frac{2\pi}{3} + \pi k$ or $\theta = \frac{7\pi}{6} + 2\pi k$, where $k \in \mathbb{Z}$." What are the four smallest positive values of $\theta$ that solve Sharon's equation?

14. Sujay solves a trigonometric equation and states "All solutions are of the form $\theta = \frac{\pi}{3} + \frac{\pi}{2}k$, where $k \in \mathbb{Z}$." Find all values of $\theta$ in the interval $-\pi \leq \theta < \pi$ that are solutions to Sujay's equation.

# Solving Trig Equations with the Pythagorean Identities (3.12) HW Solutions

1. $\csc^2\theta + \csc\theta - 2 = 0$
   $(\csc\theta + 2)(\csc\theta - 1) = 0$
   $\csc\theta + 2 = 0$ or $\csc\theta - 1 = 0$
   $\csc\theta = -2$ or $\csc\theta = 1$
   $\theta$ is a third or fourth quadrant angle with reference angle $\frac{\pi}{6}$ or $\theta$ corresponds with a point on the unit circle with a $\frac{1}{y}$ value of $1$.
   The solutions in $0 \le \theta < 2\pi$ are $\theta = \frac{7\pi}{6}$ or $\theta = \frac{11\pi}{6}$ or $\theta = \frac{\pi}{2}$.
   All solutions are of the form $\theta = \frac{7\pi}{6} + 2\pi k$ or $\theta = \frac{11\pi}{6} + 2\pi k$ or $\theta = \frac{\pi}{2} + 2\pi k$, where $k \in \mathbb{Z}$.
2. $2(1 - \cos^2\theta) = 2 + \cos\theta$
   $2 - 2\cos^2\theta = 2 + \cos\theta$
   $0 = 2\cos^2\theta + \cos\theta$
   $0 = (\cos\theta)(2\cos\theta + 1)$
   $\cos\theta = 0$ or $2\cos\theta + 1 = 0$
   $\cos\theta = 0$ or $\cos\theta = -\frac{1}{2}$
   $\theta$ corresponds with a point on the unit circle with $x$ value $0$ or $\theta$ is a second or third quadrant angle with reference angle $\frac{\pi}{3}$.
   The solutions in $0 \le \theta < 2\pi$ are $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ or $\theta = \frac{2\pi}{3}$ or $\theta = \frac{4\pi}{3}$
   All solutions are of the form $\theta = \frac{\pi}{2} + 2\pi k$ or $\theta = \frac{3\pi}{2} + 2\pi k$ or $\theta = \frac{2\pi}{3} + 2\pi k$ or $\theta = \frac{4\pi}{3} + 2\pi k$, where $k \in \mathbb{Z}$.
   This can be stated more concisely as $\theta = \frac{\pi}{2} + \pi k$ or $\theta = \frac{2\pi}{3} + 2\pi k$ or $\theta = \frac{4\pi}{3} + 2\pi k$, where $k \in \mathbb{Z}$.
3. $(1 + \tan^2\theta) + 2\tan\theta = 0$
   $\tan^2\theta + 2\tan\theta + 1 = 0$
   $(\tan\theta + 1)(\tan\theta + 1) = 0$
   $\tan\theta + 1 = 0$
   $\tan\theta = -1$
   $\theta$ is a second or fourth quadrant angle with reference angle $\frac{\pi}{4}$
   The solutions in $0 \le \theta < 2\pi$ are $\theta = \frac{3\pi}{4}$ or $\theta = \frac{7\pi}{4}$.
   All solutions are of the form $\theta = \frac{3\pi}{4} + 2\pi k$ or $\theta = \frac{7\pi}{4} + 2\pi k$, where $k \in \mathbb{Z}$.
   This can be stated more concisely as $\theta = \frac{3\pi}{4} + \pi k$, where $k \in \mathbb{Z}$.
4. $2(1 - \sin^2\theta) + 3\sin\theta - 3 = 0$
   $-2\sin^2\theta + 3\sin\theta - 1 = 0$
   $2\sin^2\theta - 3\sin\theta + 1 = 0$
   $(2\sin\theta - 1)(\sin\theta - 1) = 0$
   $2\sin\theta - 1 = 0$ or $\sin\theta - 1 = 0$
   $\sin\theta = \frac{1}{2}$ or $\sin\theta = 1$
   $\theta$ is a first or second quadrant angle with a reference angle of $\frac{\pi}{6}$ or $\theta$ corresponds with a point on the unit circle with $y$ value $1$.
   The solutions in $0 \le \theta < 2\pi$ are $\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$ or $\theta = \frac{\pi}{2}$
   All solutions to this equation are of the form $\theta = \frac{\pi}{6} + 2\pi k$ or $\theta = \frac{5\pi}{6} + 2\pi k$ or $\theta = \frac{\pi}{2} + 2\pi k$, where $k \in \mathbb{Z}$.
5. $(1 + \cot^2\theta) + \cot\theta = 1$
   $\cot^2\theta + \cot\theta = 0$
   $(\cot\theta)(\cot\theta + 1) = 0$
   $\cot\theta = 0$ or $\cot\theta + 1 = 0$

$\cot\theta = 0$ or $\cot\theta = -1$

$\theta$ corresponds the points on the unit circle with $\frac{x}{y}$ values of $0$ (i.e. an $x$ value of $0$) or $\theta$ is a second or fourth quadrant angle with reference angle $\frac{\pi}{4}$.

The solutions in $0 \leq \theta < 2\pi$ are $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ or $\theta = \frac{3\pi}{4}$ or $\theta = \frac{7\pi}{4}$

All solutions to this equation are of the form $\theta = \frac{\pi}{2} + 2\pi k$ or $\theta = \frac{3\pi}{2} + 2\pi k$ or $\theta = \frac{3\pi}{4} + 2\pi k$ or $\theta = \frac{7\pi}{4} + 2\pi k$, where $k \in \mathbb{Z}$.

This can be stated more concisely as $\theta = \frac{\pi}{2} + \pi k$ or $\theta = \frac{3\pi}{4} + \pi k$, where $k \in \mathbb{Z}$.

6. $\sqrt{2}\cos\theta\sin\theta - \sin\theta = 0$

   $(\sin\theta)\left(\sqrt{2}\cos\theta - 1\right) = 0$

   $\sin\theta = 0$ or $\sqrt{2}\cos\theta - 1 = 0$

   $\sin\theta = 0$ or $\cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

   $\theta$ corresponds with a point on the unit circle with $y$ value $0$ or $\theta$ is a first or fourth quadrant angle with reference angle $\frac{\pi}{4}$.

   The solutions in $0 \leq \theta < 2\pi$ are $\theta = 0$ or $\theta = \pi$ or $\theta = \frac{\pi}{4}$ or $\theta = \frac{7\pi}{4}$.

   All solutions to this equation are of the form $\theta = 0 + 2\pi k$ or $\theta = \pi + 2\pi k$ or $\theta = \frac{\pi}{4} + 2\pi k$ or $\theta = \frac{7\pi}{4} + 2\pi k$, where $k \in \mathbb{Z}$.

   This can be stated more concisely as $\theta = \pi k$ or $\theta = \frac{\pi}{4} + 2\pi k$ or $\theta = \frac{7\pi}{4} + 2\pi k$, where $k \in \mathbb{Z}$.

7. $\sin^2\theta + \cos\theta + 1 = 0$

   $(1 - \cos^2\theta) + \cos\theta + 1 = 0$

   $-\cos^2\theta + \cos\theta + 2 = 0$

   $\cos^2\theta - \cos\theta - 2 = 0$

   $(\cos\theta + 1)(\cos\theta - 2) = 0$

   $\cos\theta + 1 = 0$ or $\cos\theta - 2 = 0$

   $\cos\theta = -1$ or $\cos\theta = 2$

   $\cos\theta = 2$ is impossible as the output of cosine will never be lower than $-1$ ($x$-values on the unit circle are never lower than $-1$).

   $\cos\theta = -1$

   $\theta$ corresponds with a point on the unit circle with $x$ value $-1$

   The only zero in the interval $0 \leq \theta < 2\pi$ is $\theta = \pi$

   All zeros of this function are of the form $\theta = \pi + 2\pi k$, where $k \in \mathbb{Z}$.

8. $1 + \tan^2\theta = 0$

   $\sec^2\theta = 0$

   $(\sec\theta)(\sec\theta) = 0$

   $\sec\theta = 0$

   This is impossible as secant can never have an output value of zero ($\frac{1}{x}$ is never going to be zero).

   The given function has no zeros.

9. $\sec^2 x = 1$

   $\sec^2 x - 1 = 0$

   $(\sec x - 1)(\sec x + 1) = 0$

   $\sec x - 1 = 0$ or $\sec x + 1 = 0$

   $\sec x = 1$ or $\sec x = -1$

   $x$ corresponds with a point on the unit circle with $\frac{1}{x}$ value $1$ or $x$ corresponds with a point on the unit circle with $\frac{1}{x}$ value $-1$.

   $x = 0 + 2\pi k$ or $x = \pi + 2\pi k$, where $k \in \mathbb{Z}$.

This can be stated more concisely as $x = \pi k$, where $k \in \mathbb{Z}$.

10. $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2$
    $(\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta) + (\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta)$
    $(\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta) + (\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta)$
    $(1 + 2\sin\theta\cos\theta) + (1 - 2\sin\theta\cos\theta)$
    $1 + 1 + 2\sin\theta\cos\theta - 2\sin\theta\cos\theta$
    $$= \mathbf{2}$$
11. $(2 + \tan\theta)^2 + (1 - 2\tan\theta)^2$
    $(4 + 4\tan\theta + \tan^2\theta) + (1 - 4\tan\theta + 4\tan^2\theta)$
    $5 + 5\tan^2\theta$
    $5(1 + \tan^2\theta)$
    $$= \mathbf{5\sec^2\theta}$$
12. $(2 - \csc\theta)(2 + \csc\theta)$
    $4 - \csc^2\theta$
    $4 - (1 + \cot^2\theta)$
    $4 - 1 - \cot^2\theta$
    $$= \mathbf{3 - \cot^2\theta}$$
13. $\theta = \frac{2\pi}{3} + \pi k$ represents the solution set $\theta \in \left\{\dots, -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}, \dots\right\}$
    $\theta = \frac{7\pi}{6} + 2\pi k$ represents the solution set $\theta \in \left\{\dots - \frac{17\pi}{6}, -\frac{5\pi}{6}, \frac{7\pi}{6}, \frac{19\pi}{6}, \frac{31\pi}{6}, \dots\right\}$
    The four smallest positive solutions are $\left\{\frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}, \frac{8\pi}{3}\right\}$
14. $\theta = \frac{\pi}{3} + \frac{\pi}{2}(2) = \frac{\pi}{3} + \pi$ is outside the given interval
    $\theta = \frac{\pi}{3} + \frac{\pi}{2}(1) = \frac{\pi}{3} + \frac{\pi}{2} = \frac{2\pi}{6} + \frac{3\pi}{6} = \frac{5\pi}{6}$
    $\theta = \frac{\pi}{3} + \frac{\pi}{2}(0) = \frac{\pi}{3}$
    $\theta = \frac{\pi}{3} + \frac{\pi}{2}(-1) = \frac{\pi}{3} - \frac{\pi}{2} = \frac{2\pi}{6} - \frac{3\pi}{6} = -\frac{\pi}{6}$
    $\theta = \frac{\pi}{3} + \frac{\pi}{2}(-2) = \frac{\pi}{3} - \pi = \frac{\pi}{3} - \frac{3\pi}{3} = -\frac{2\pi}{3}$
    $\theta = \frac{\pi}{3} + \frac{\pi}{2}(-3) = \frac{\pi}{3} - \frac{3\pi}{2} = \frac{2\pi}{6} - \frac{9\pi}{6} = -\frac{7\pi}{6}$ is outside the given interval
    The solutions inside the interval $-\pi \le \theta < \pi$ are $-\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}$, and $\frac{5\pi}{6}$.